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Boundedness of Solutions of Difference Equations and Application to Numerical Solution of Volterra Integral Equations of the Second Kind*

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We study the difference equations obtained when some numerical methods for Volterra integral equations of the second kind are applied to the linear test problem $y(t)=1+\int_0^t(\lambda+\mu t+\nu s)\,y(s)\,ds$, $t\geqslant 0$, with fixed stepsize h. The resulting difference equations are of Poincaré type and we formulate a criterion for boundedness of solutions of these equations if the associated characteristic polynomial is a simple von Neumann polynomial. This result is then used in stability analysis of reducible quadrature methods for Volterra integral equations. © 1986 Academic Press, Inc

1. INTRODUCTION

Consider the difference equation of fixed order r with variable coefficients

$$\sum_{i=0}^{r} \alpha_{i,n}\, y_{n-i}=0, \tag{1}$$

$n=r,\ r+1,\ldots$, where $\alpha_{i,n}=\alpha_i+n^{-1}\beta_i$, $i=0,1,\ldots,r$, and where α_i and β_i are constants. As explained in Section 3 such equations arise when some numerical methods for Volterra integral equations (VIEs) of the second kind are applied to the scalar test equation

$$y(t)=1+\int_0^t (\lambda+\mu t+\nu s)\, y(s)\, ds, \qquad t\geqslant 0,$$

with fixed stepsize $h>0$. Denote by ϕ and ψ the polynomials

$$\phi(\xi)=\sum_{i=0}^{r} \alpha_i \xi^{r-i}, \qquad \psi(\xi)=\sum_{i=0}^{r} \beta_i \xi^{r-i}, \tag{2}$$

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which are called the characteristic polynomials associated with (1). This equation is a special case of difference equations with "almost constant coefficients" also called equations of Poincaré type (see [11–14]). One of the main facts about such equations is the Perron theorem [12] which we state here as given in Milne-Thomson [11, p. 548].

THEOREM 1. *Let $q_1, q_2, ..., q_s$ be the distinct moduli of the roots of the polynomial ϕ and let l_λ be the number of roots whose modulus is q_λ, multiple roots being counted according to their multiplicity, so that $l_1 + l_2 + \cdots + l_s = r$. Then, provided that $\alpha_0 \neq 0$ and $\alpha_{r,n} \neq 0$ for all values of n, the difference equation (1) has a fundamental system of solutions, which fall into s classes, such that, for the solutions of the λth class and their linear combinations,*

$$\limsup_{n \to \infty} \sqrt[n]{|y_n|} = q_\lambda.$$

The number of solutions of the λth class is l_λ.

As an immediate consequence of this theorem it follows that if all roots of the polynomial ϕ have modulus less than one, then every solution of (1) is bounded. Similarly, if ϕ has a root with modulus greater than one than there exists an unbounded solution of (1). Unfortunately, this theorem cannot be used to decide whether or not the solutions of (1) are bounded if ϕ has some roots with modulus equal to one. This can be the case in applications in numerical analysis (see [1]). It is the purpose of this paper to examine this question. To be more precise, we will establish a criterion for boundedness of solutions of (1) if ϕ is a simple von Neumann polynomial. This means that all roots of ϕ have modulus less than or equal to one and those with modulus one are simple (compare the definition of this notion in [10]). This criterion is then used in stability analysis of quadrature methods reducible to linear multistep methods for Volterra integral equations of the second kind. The results obtained here extend and strengthen some results of our previous paper [1].

2. BOUNDEDNESS OF SOLUTIONS OF DIFFERENCE EQUATIONS

Assume that ϕ is a simple von Neumann polynomial and denote its essential roots, i.e., those with modulus equal to one, by $\xi_1, \xi_2, ..., \xi_k$. Denote the nonessential roots of ϕ, i.e., those with modulus less than one, by $\xi_{k+1}, \xi_{k+2}, ..., \xi_s$, and their multiplicities by $m_{k+1}, m_{k+2}, ..., m_s$, so that $k + m_{k+1} + m_{k+2} + \cdots + m_s = r$. Define the family of polynomials Φ_n by

$$\Phi_n(\xi) = \phi(\xi) + n^{-1}\psi(\xi),$$

$n = 1, 2, \ldots$, and denote the roots of Φ_n by

$$\xi_{1,n}, \ldots, \xi_{k,n}, \xi^1_{k+1,n}, \ldots, \xi^{m_{k+1}}_{k+1,n}, \ldots, \xi^1_{s,n}, \ldots, \xi^{m_s}_{s,n}, \tag{3}$$

where

$$\begin{aligned} &\xi_{i,n} \to \xi_i, \qquad i = 1, 2, \ldots, k, \\ &\xi^j_{i,n} \to \xi_i, \qquad j = 1, 2, \ldots, m_i,\ i = k+1, k+2, \ldots, s, \end{aligned}$$

as $n \to \infty$. Let us compute the $O(n^{-1})$ approximations to the roots $\xi_{i,n}$ approaching essential roots of ϕ. Put

$$\xi_{i,n} = \xi_i + \gamma_i n^{-1} + O(n^{-2}), \tag{4}$$

$i = 1, 2, \ldots, k$. Since

$$\phi(\xi_{i,n}) = \phi'(\xi_i)\, \gamma_i n^{-1} + O(n^{-2})$$

and

$$\psi(\xi_{i,n}) = \psi(\xi_i) + O(n^{-2}),$$

it follows that

$$\Phi_n(\xi_{i,n}) = (\phi'(\xi_i)\, \gamma_i + \psi(\xi_i))\, n^{-1} + O(n^{-2}) = 0.$$

Hence,

$$\gamma_i = -\psi(\xi_i)/\phi'(\xi_i),$$

$i = 1, 2, \ldots, k$. For the roots $\xi^j_{i,n}$ approaching the nonessential root ξ_i of ϕ of multiplicity m_i we have

$$\xi^j_{i,n} = \xi_i + \sum_{l=1}^{m_i} \delta^j_{i,l} n^{-l/m_i} + O(n^{-(m_i+1)/m_i}), \tag{5}$$

$j = 1, 2, \ldots, m_i$, $i = k+1, k+2, \ldots, s$ (compare the discussion in [6, p. 236]). Let us compute the first terms $\delta^j_{i,1}$ in these expressions. In view of the relations

$$\begin{aligned} &\phi(\xi^j_{i,n}) = \phi^{(m)}(\xi_i)(\delta^j_{i,1})^{m_i} n^{-1}/m_i! + O(n^{-(m_i+1)/m_i}), \\ &\psi(\xi^j_{i,n}) = \psi(\xi_i) + O(n^{-1/m_i}), \end{aligned}$$

and $\phi^{(m)}(\xi_i)/m_i! = a_0$, it follows that

$$\Phi_n(\xi^j_{i,n}) = (a_0(\delta^j_{i,1})^{m_i} + \psi(\xi_i))\, n^{-1} + O(n^{-(m_i+1)/m_i})$$

as $n \to \infty$. Thus if ϕ and ψ have no common factor it follows from $\psi(\xi_i) \neq 0$ and $a_0 \neq 0$ that $\delta_{i,1}^j$, $j = 1, 2,..., m_i$, are m_i different values of $(\psi(\xi_i)/a_0)^{1/m_i}$. Consequently, all roots of Φ_n are distinct for large n.

Occasionally, it will be convenient to denote the sequence (3) by

$$\lambda_{1,n},..., \lambda_{k,n}, \lambda_{k+1,n},..., \lambda_{r,n}. \tag{6}$$

Define

$$\bar{y}_n = [y_n, y_{n-1},..., y_{n-r+1}]^{\mathrm{T}}$$

and

$$A_n = \begin{bmatrix} -\dfrac{\alpha_{1,n}}{\alpha_{0,n}} & -\dfrac{\alpha_{2,n}}{\alpha_{0,n}} & \cdots & -\dfrac{\alpha_{r,n}}{\alpha_{0,n}} \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

where T stands for transposition. Then Eq. (1) can be written in vector form

$$\bar{y}_n = A_n \bar{y}_{n-1}, \tag{7}$$

$n = r, r+1,...$. The characteristic polynomial $f_n = \det(A_n - \lambda I)$ of A_n is then

$$f_n(\lambda) = (-1)^r \sum_{j=0}^{r} (\alpha_{j,n}/\alpha_{0,n}) \lambda^{r-j} = (-1)^r \Phi_n(\lambda)/\alpha_{0,n}$$

and its roots are given by (6). Since these roots are distinct, it follows that

$$P_n^{-1} A_n P_n = D_n, \tag{8}$$

where P_n is the Vandermonde matrix

$$P_n = \begin{bmatrix} \lambda_{1,n}^{r-1} & \lambda_{2,n}^{r-1} & \cdots & \lambda_{r,n}^{r-1} \\ \vdots & \vdots & & \vdots \\ \lambda_{1,n} & \lambda_{2,n} & \cdots & \lambda_{r,n} \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

and

$$D_n = \mathrm{diag}[\lambda_{1,n}, \lambda_{2,n},..., \lambda_{r,n}].$$

We will need the following result concerning the matrix P_n^{-1}.

LEMMA 1. *Assume that for* $i=1,2,..,k$, $\xi_{i,n}$ *satisfies* (4) *and that for* $j=1,2,...,m_i$, $i=k+1, k+2,..., s$, $\xi^j_{i,n}$ *are as in* (3). *Then there exists a number* $v\in[0,1)$ *such that* $\|P_n^{-1}\|_\infty = O(n^v)$ *as* $n\to\infty$.

Proof. Define

$$L_{i,n}(t)=(t-\lambda_{1,n})\cdots(t-\lambda_{i-1,n})(t-\lambda_{i+1,n})\cdots(t-\lambda_{r,n})/D_{i,n},$$

where

$$D_{i,n}=(\lambda_{i,n}-\lambda_{1,n})\cdots(\lambda_{i,n}-\lambda_{i-1,n})(\lambda_{i,n}-\lambda_{i+1,n})\cdots(\lambda_{i,n}-\lambda_{r,n}),$$

$i=1,2,..., r$. It is well known that the (i,j)-element of the inverse matrix P_n^{-1} is the coefficient of t^{r-j} in the polynomial $L_{i,n}$ (compare [4]). Hence, putting

$$P_n^{-1}=[p^*_{i,j,n}]_{i,j=1,2,...,r},$$

it follows that $p^*_{i,j,n}=\sigma_{i,j,n}/D_{i,n}$, where the numbers $\sigma_{i,j,n}$ are bounded uniformly in n. It is clear that $p^*_{i,j,n}=O(1)$ for $i=1,2,..., k$. To estimate $p^*_{i,j,n}$ for $i=k+1, k+2,..., r$, note that $\lambda_{i,n}=\xi^v_{\mu,n}$ for some $\mu\in\{k+1, k+2,..., s\}$ and $v\in\{1,2,..., m_\mu\}$. Hence, the denominator $D_{i,n}$ of $p^*_{i,j,n}$ contains the product

$$(\xi^1_{\mu,n}-\xi^2_{\mu,n})\cdots(\xi^1_{\mu,n}-\xi^{m_\mu}_{\mu,n})$$

for $v=1$; the product

$$(\xi^v_{\mu,n}-\xi^1_{\mu,n})\cdots(\xi^v_{\mu,n}-\xi^{v-1}_{\mu,n})(\xi^v_{\mu,n}-\xi^{v+1}_{\mu,n})\cdots(\xi^v_{\mu,n}-\xi^{m_\mu}_{\mu,n})$$

for $v=2,3,..., m_\mu-1$; and the product

$$(\xi^{m_\mu}_{\mu,n}-\xi^1_{\mu,n})\cdots(\xi^{m_\mu}_{\mu,n}-\xi^{m_\mu-1}_{\mu,n})$$

for $v=m_\mu$, while the other factors are bounded away from zero, uniformly with respect to n. Since $\xi^\rho_{\mu,n}-\xi^\tau_{\mu,n}=O(n^{1/m_\mu})$ for $\rho,\tau\in\{1,2,..., m_\mu\}$ it follows that $D_{i,n}=O(n^{-(m_\mu-1)/m_\mu})$ and $p^*_{i,j,n}=O(n^{(m_\mu-1)/m_\mu})$, $i=k+1, k+2,..., r$, as $n\to\infty$. Consequently, there exists a number $v\in[0,1)$ such that $\|P_n^{-1}\|=O(n^v)$ as $n\to\infty$, which is our claim.

We will also need a bound on the norm of the matrix $\Omega_n := P_{n-1}-P_n$.

LEMMA 2. *Under the same assumptions as in Lemma* 1, *there exists a* $u\in(1,2]$ *such that* $\|\Omega_n\|_\infty=O(n^{-u})$ *as* $n\to\infty$.

Proof. Let $\Omega_n=[\omega_{i,j,n}]_{i,j=1,2,...,r}$. To estimate $\omega_{i,j,n}$ we first estimate the differences $\lambda_{i,n-1}-\lambda_{i,n}$, $i=1,2,..., r$. Using (4) it follows that

$$\lambda_{i,n-1}-\lambda_{i,n}=O(n^{-2}),\qquad i=1,2,..., k.$$

For $i=k+1, k+2,\dots, r$, there exist $\mu\in\{k+1, k+2,\dots, s\}$ and $\nu\in\{1, 2,\dots, m_\mu\}$ such that $\lambda_{i,n}=\xi^\nu_{\mu,n}$. Hence, in view of (5) we obtain

$$\xi^\nu_{\mu,n-1}-\xi^\nu_{\mu,n}=\sum_{l=1}^{m_\mu}\delta^\nu_{\mu,l}((n-1)^{-l/m_\mu}-n^{-l/m_\mu})+O(n^{-(m_\mu+1)/m_\mu}).$$

Clearly,

$$(n-1)^{-l/m_\mu}-n^{-l/m_\mu}=O(n^{-(m_\mu+l)/m_\mu}),$$

$l=1, 2,\dots, m_\mu$, hence

$$\xi^\nu_{\mu,n-1}-\xi^\nu_{\mu,n}=O(n^{-(m_\mu+1)/m_\mu}),\qquad \nu=1, 2,\dots, m_\mu,\ \mu=k+1, k+2,\dots, s.$$

Now it is clear that there exist a $u\in(1, 2]$ such that

$$\lambda_{i,n-1}-\lambda_{i,n}=O(n^{-u}),\qquad i=1, 2,\dots, r,$$

and, consequently, $\|\Omega_n\|_\infty=O(n^{-u})$, which is the desired estimate.

In light of the above preliminary results we are now ready to formulate an prove our main theorem.

THEOREM 2. *Assume that* $\alpha_0\neq 0$ *and* $\alpha_{0,n}\neq 0$ *for all* n; *the polynomials* ϕ *and* ψ *have no common factor and* ϕ *is a simple von Neumann polynomial with essential roots* $\xi_1, \xi_2,\dots, \xi_k$ ($k=0$ *is allowed*). *Then every solution of* (1) *is bounded, provided that*

$$\tfrac{1}{2}\pi\leqslant|\mathrm{Arg}(\xi_i)-\mathrm{Arg}(\gamma_i)|\leqslant\tfrac{3}{2}\pi,\tag{9}$$

where $\gamma_i=-\psi(\xi_i)/\phi'(\xi_i)$, $i=1, 2,\dots, k$. *Here,* $\mathrm{Arg}(z)$ *stands for the principal value of the argument of the complex number* z, *i.e.,* $\mathrm{Arg}(z)\in(-\pi, \pi]$.

Proof. In view of (8), the solution of the system (7), which is equivalent to (1), is

$$\bar y_n=P_nD_n\prod_{m=r+1}^{n}(P_m^{-1}P_{m-1}D_{m-1})P_r^{-1}\bar y_{r-1},\tag{10}$$

$n=r, r+1,\dots$, *where* $\bar y_{r-1}$ *is a given starting vector. Hence,*

$$\|\bar y_n\|_\infty\leqslant\|P_n\|_\infty\,\|D_n\|_\infty\prod_{m=r+1}^{n}\|P_m^{-1}P_{m-1}D_{m-1}\|_\infty\,\|P_r^{-1}\|_\infty\,\|\bar y_{r-1}\|_\infty,\tag{11}$$

$n=r, r+1,\dots$. Taking into account the form of the inverse P_m^{-1} of the Vandermonde matrix P_m (see [4]) and the definition of D_m, it is easy to check that

$$P_m^{-1} P_{m-1} D_{m-1} = [\lambda_{j,m-1} L_{i,m}(\lambda_{j,m-1})]_{i,j=1,2,\dots,r}.$$

It follows from Lemmas 1 and 2 that

$$\|P_m^{-1}\Omega_m\|_\infty \leqslant \|P_m^{-1}\|_\infty \, \|\Omega_m\|_\infty = O(m^{v-u}),$$

where $v-u<0$. Thus, $P_m^{-1}\Omega_m \to 0$ (zero matrix) as $m\to\infty$ and $P_m^{-1}P_{m-1} = I + P_m^{-1}\Omega_m \to I$ (identity matrix) as $m\to\infty$. Similarly,

$$P_m^{-1} P_{m-1} D_{m-1} \to D, \qquad m\to\infty, \tag{12}$$

where

$$D = \operatorname{diag}[\xi_1,\dots,\xi_k, \underbrace{\xi_{k+1},\dots,\xi_{k+1}}_{m_{k+1}},\dots, \underbrace{\xi_s,\dots,\xi_s}_{m_s}].$$

Consider now the norm

$$\|P_m^{-1}P_{m-1}D_{m-1}\|_\infty = \max_{1\leqslant i\leqslant r} \sum_{j=1}^{r} |\lambda_{j,m-1}| \, |L_{i,m}(\lambda_{j,m-1})|.$$

In view of (12) and the fact that $\xi_{k+1}, \xi_{k+2},\dots, \xi_s$ are nonessential roots of ϕ, it follows that for sufficiently large m this maximum is attained for some i such that $1\leqslant i\leqslant k$, i.e.,

$$\|P_m^{-1}P_{m-1}D_{m-1}\|_\infty = \max_{1\leqslant i\leqslant k} \left\{ |\lambda_{i,m-1}| \, |L_{i,m}(\lambda_{i,m-1})| + \sum_{\substack{j=1\\ j\neq i}}^{r} |\lambda_{j,m-1}| \, |L_{i,m}(\lambda_{j,m-1})| \right\} \tag{13}$$

(for $k=0$ the theorem follows immediately from Theorem 1). Using (4), it is clear that

$$\frac{\lambda_{i,m-1} - \lambda_{l,m}}{\lambda_{i,m} - \lambda_{l,m}} = 1 + O(m^{-2})$$

for $i=1,2,\dots,k$, and $l\neq i$. Hence,

$$L_{i,m}(\lambda_{i,m-1}) = 1 + O(m^{-2}), \qquad i=1,2,\dots,k.$$

To estimate $L_{i,m}(\lambda_{j,m-1})$ for $i=1,2,\dots,k$, and $j\neq i$ note that the denominator $D_{i,m}$ of $L_{i,m}(\lambda_{j,m-1})$ is bounded away from zero while the numerator of $L_{i,m}(\lambda_{j,m-1})$ contains either the term $\lambda_{j,m-1}-\lambda_{j,m}$ if $j<i$, or the product

$$(\xi^{\nu}_{\mu,m-1}-\xi^{1}_{\mu,m})\cdots(\xi^{\nu}_{\mu,m-1}-\xi^{m_\mu}_{\mu,m})$$

for some $\mu\in\{k+1,k+2,\dots,s\}$ and $\nu\in\{1,2,\dots,m_\mu\}$ if $j>i$. Therefore, since the relations

$$\begin{aligned}
&\lambda_{j,m-1}-\lambda_{j,m}=O(m^{-2}), && j>i,\\
&\xi^{\nu}_{\mu,m-1}-\xi^{\nu}_{\mu,m}=O(m^{-(m_\mu+1)/m_\mu}) && \text{(see the proof of Lemma 1)},\\
&\xi^{\nu}_{\mu,m-1}-\xi^{\tau}_{\mu,m}=O(m^{-1/m_\mu}), && \tau\neq\nu
\end{aligned}$$

are satisfied, it follows that

$$L_{i,m}(\lambda_{j,m-1})=O(m^{-2}),\qquad i=1,2,\dots,k,\ j\neq i.$$

For $|\lambda_{i,m-1}|$, $i=1,2,\dots,k$ we have

$$|\lambda_{i,m-1}|=|\xi_i+\gamma_i(m-1)^{-1}+O(m^{-2})|,$$

where $|\xi_i|=1$. Hence, if the condition (9) given in the formulation of our theorem is satisfied then $|\lambda_{i,m-1}|=1+O(m^{-2})$. To see this suppose that ξ_i and γ_i are perpendicular, which corresponds to the exteme case in (9). Then $\xi_i=\exp(i\theta)$, $\gamma_i=|\gamma_i|\exp(i(\theta\pm\pi/2))$, and elementary computations show that $|\xi_i+\gamma_i(m-1)^{-1}|=1+O(m^{-2})$ as $m\to\infty$. In view of (13), it follows that

$$\|P_m^{-1}P_{m-1}D_{m-1}\|_\infty=1+O(m^{-2}),\qquad m\to\infty.$$

It is clear that $\|P_n\|_\infty$ and $\|D_n\|_\infty$ are bounded uniformly in n. Hence, because the infinite product $\prod_{m=r+1}^{\infty}(1+O(m^{-2}))$ is convergent, the assertion of our theorem follows from (11).

Remark 1. Because Eq. (1) includes as special case case the difference equation with constant coefficients ($\beta_i=0$, $i=0,1,\dots,r$), the condition that ϕ is a simple von Neumann polynomial cannot be relaxed to the condition that ϕ is a von Neumann polynomial, i.e., a polynomial with all roots in the closed unit disc, where the essential roots are not necessarily distinct.

Remark 2. Condition (9) is essential. To illustrate this consider the equation

$$y_n-(1-n^{-1}\beta_1)\,y_{n-1}=0,$$

$n=1, 2, \ldots$, $y_0=1$, with $\phi(\xi)=\xi-1$, $\psi(\xi)=\beta_1$, where β_1 is real. Now $\xi_1=1$, $\gamma_1=-\beta_1$, and the condition (9) is equivalent to $\beta_1>0$. Observe also that if $\beta_1<0$ then the solution $y_n=\prod_{m=1}^{n}(1-m^{-1}\beta_1)$ of this equation is unbounded.

Remark 3. If ϕ and ψ have a common factor $\xi-\xi^*$ then Eq. (1) with given starting values $y_0, y_1, \ldots, y_{r-1}$, is equivalent to the equation

$$\sum_{i=0}^{r-1} (\alpha_i^*+n^{-1}\beta_i^*)\, z_{n-i}=0, \tag{14}$$

$n=r, r+1, \ldots$, where $z_n=y_n-\xi^* y_{n-1}$, $n=1, 2, \ldots$, and where α_i^* and β_i^*, $i=0, 1, \ldots, r-1$, are defined by

$$\sum_{i=0}^{r} \alpha_i \xi^{r-i}=(\xi-\xi^*) \sum_{i=0}^{r-1} \alpha_i^* \xi^{r-1-i},$$

$$\sum_{i=0}^{r} \beta_i \xi^{r-i}=(\xi-\xi^*) \sum_{i=0}^{r-1} \beta_i^* \xi^{r-1-i}.$$

Nothing essential is lost if $|\xi^*|<1$ because in this case the solution y_n of (1) is bounded if the solution z_n of (14) is bounded. However, if $|\xi^*|=1$ it may happen that z_n is bounded while y_n is not.

Remark 4. The assumption $\alpha_{0,n}\neq 0$ for all n could be replaced by the requirement that $\alpha_{0,n}\neq 0$ for all sufficiently large n. The proof would require only slight modifications.

Remark 5. Interesting results about the asymptotic behaviour of solutions of equations of type (1) with $\alpha_{i,n}=\alpha_i+\beta_{i,n}$, under the condition $\sum_{n=r}^{\infty}\beta_{i,n}<\infty$, were obtained by Evgrafov [7] and Coffman [5]. This condition is, of course, not satisfied for $\alpha_{i,n}=\alpha_i+n^{-}\beta_i$, which is the case under consideration.

3. Stability Analysis of Reducible Quadrature Methods for Volterra Integral Equations of the Second Kind

Denote by ρ and σ the characteristic polynomials of linear multistep method for ordinary differential equations (ODEs) with coefficients a_i and b_i, $i=0, 1, \ldots, k$, $a_0\neq 0$, $|a_k|+|b_k|\neq 0$, i.e.,

$$\rho(\xi)=\sum_{i=0}^{k} a_i \xi^{k-i}, \qquad \sigma(\xi)=\sum_{i=0}^{k} b_i \xi^{k-i}. \tag{15}$$

It is always assumed that the linear multistep method is:

(i) consistent, i.e., $\rho(1)=0$ and $\rho'(1)=\sigma(1)$.
(ii) zero stable, i.e., ρ is a simple von Neumann polynomial.
(iii) implicit, i.e., $b_0 \neq 0$.

Moreover, it is assumed that:

(iv) the polynomials ρ and σ have no common factor.

The last assumption is made for the sake of convenience but nothing essential is lost (compare the discussion of this condition in [6]).

Consider the Volterra integral equation of the second kind

$$y(t)=g(t)+\int_0^t k(t, s, y(s))\,ds, \qquad t \geqslant 0, \tag{16}$$

with the solution Y. Suppose the stepsize $h>0$ is fixed and define the grid $t_i=ih$, $i=0, 1,...$. The direct quadrature method for (16) is the method of the form

$$y_n=g(t_n)+h\sum_{j=0}^{n} w_{n,j}k(t_n, t_j, y_j), \tag{17}$$

$n=0, 1,...,$ where y_n is an approximation to $Y(t_n)$. This method is said to be (ρ, σ)-reducible if the weights $w_{n,j}$ are associated with the coefficients a_i and b_i of linear multistep method for ODEs by the formulas

$$\begin{aligned}\sum_{i=0}^{k} a_i w_{n-i,j} &= 0, \qquad && j=0, 1,..., n-k-1,\\ &= b_{n-j}, \qquad && j=n-k, n-k+1,..., n.\end{aligned} \tag{18}$$

These methods have been studied by Wolkenfelt [16–18] and van der Houwen and Wolkenfelt [15] (see also [1–3, 9]). As in [1] we study the stability properties of reducible quadrature methods applying them to the linear test equation

$$y(t)=1+\int_0^t (\lambda+\mu t+\nu s)\,y(s)\,ds, \qquad t \geqslant 0, \tag{19}$$

with fixed stepsize $h>0$. It is known (see [1]) that the solution of this equation is bounded if $\mu+\nu<0$ and $2\mu+\nu<0$ and we investigate whether this property is inherited by the numerical solution approximating Y. It was shown in [1] that the application of (17) subject to (18) to this equation leads to the difference equation with variable coefficients

$$\sum_{l=0}^{k} \sum_{i=0}^{k} a_l[a_i - b_i(h\lambda + h^2\mu(n-2l) + h^2\nu(n-i-l))]\, y_{n-i-l} = 0,$$

$n = 2k, 2k+1, \ldots$. This relation is equivalent to

$$h^2(\mu+\nu) \sum_{l=0}^{k} \sum_{i=0}^{k} a_l b_i\, y_{n-i-l}$$
$$-n^{-1} \sum_{l=0}^{k} \sum_{i=0}^{k} a_l[a_i - b_i(h\lambda - 2h^2\mu l - h^2\nu(i+l))]\, y_{n-i-l} = 0. \tag{20}$$

The reducible quadrature method is said to be absolutely stable for given $h\lambda$, $h^2\mu$, and $h^2\nu$ if, for these values, every solution y_n of (20) is bounded. A region $\mathscr{A}$ in $(h\lambda, h^2\mu, h^2\nu)$-space is said to be the region of absolute stability of the method (17) if (17) is absolutely stable for all $(h\lambda, h^2\mu, h^2\nu) \in \mathscr{A}$.

Consider the family of polynomials associated with Eq. (20)

$$\Phi_n(\xi) = \phi(\xi) + n^{-1}\psi(\xi),$$

$n = 2k, 2k+1, \ldots$, where

$$\phi(\xi) = h^2(\mu+\nu) \sum_{l=0}^{k} \sum_{i=0}^{k} a_l b_i \xi^{2k-i-l},$$

$$\psi(\xi) = -\sum_{l=0}^{k} \sum_{i=0}^{k} a_l[a_i - b_i(h\lambda - 2h^2\mu l - h^2\nu(i+l))]\, \xi^{2k-i-l}.$$

It is clear that

$$\phi(\xi) = h^2(\mu+\nu)\, \rho(\xi)\, \sigma(\xi),$$

and using the relations

$$\sum_{l=0}^{k} l a_l \xi^{k-l} = k\rho(\xi) - \xi\rho'(\xi),$$

$$\sum_{i=0}^{k} i b_i \xi^{k-i} = k\sigma(\xi) - \xi\sigma'(\xi),$$

we also have

$$\psi(\xi) = -\rho^2(\xi) + h\lambda\rho(\xi)\,\sigma(\xi) - h^2(2\mu+\nu)\,\sigma(\xi)(k\rho(\xi) - \xi\rho'(\xi))$$
$$-h^2\nu\rho(\xi)(k\sigma(\xi) - \xi\sigma'(\xi)).$$

Assume that σ is a simple von Neumann polynomial. Then using condition (iv) it is clear that the polynomials ϕ and ψ do not have a common root with modulus equal to one, although it may happen that they have a common root inside the unit circle. Denote by $\xi_1 = 1$, $\xi_2, \ldots, \xi_p$, the essential roots of ρ and by $\xi_{p+1}, \xi_{p+2}, \ldots, \xi_{p+q}$ the essential roots of σ and define

$$\gamma_i = -\psi(\xi_i)/\phi'(\xi_i), \qquad i = 1, 2, \ldots, p+q.$$

It is easy to check that

$$\gamma_i = -\xi_i \frac{2\mu + v}{\mu + v} \tag{21}$$

for $i = 1, 2, \ldots, p$, and

$$\gamma_i = \frac{\rho(\xi_i) - h^2 v \xi_i \sigma'(\xi_i)}{h^2(\mu + v)\,\sigma'(\xi_i)} \tag{22}$$

for $i = p+1, p+2, \ldots, p+q$. As an immediate consequence of Theorem 2 and Remark 3 we have

THEOREM 3. *Assume that* $\mu \neq v$; *the conditions* (i) – (iv) *are satisfied and that* σ *is a simple von Neumann polynomial. Then the region of absolute stability of the* (ρ, σ)*-reducible quadrature method* (17) *is given by*

$$\mathscr{A} = \{(h\lambda, h^2\mu, h^2 v) : \tfrac{1}{2}\pi \leqslant |\mathrm{Arg}(\xi_i) - \mathrm{Arg}(\gamma_i)| \leqslant \tfrac{3}{2}\pi,\ i = 1, 2, \ldots, p+q\},$$

where $\xi_1 = 1, \xi_2, \ldots, \xi_p$ *are the essential roots of* ρ, $\xi_{p+1}, \xi_{p+2}, \ldots, \xi_{p+q}$ *are the essential roots of* σ, *and where* γ_i *are defined by* (21) *for* $i = 1, 2, \ldots, p$ *and by* (22) *for* $i = p+1, p+2, \ldots, p+q$.

It follows from this theorem that if $\xi_1 = 1$ is the only essential root of ρ and if all roots of σ are inside the unit disc then the region of absolute stability of such method is $\mathscr{A} = \mathscr{B} \cup \mathscr{C}$, *where*

$$\mathscr{B} = \{(h\lambda, h^2\mu, h^2 v) : 2\mu + v < 0,\ \mu + v < 0\},$$
$$\mathscr{C} = \{(h\lambda, h^2\mu, h^2 v) : 2\mu + v > 0,\ \mu + v > 0\}.$$

Examples of such methods are given by quadrature methods reducible to backward differentiation formulas for ODEs for $k = 1, 2, \ldots, 6$.

Consider now the θ-method for VIEs examined in [1]:

$$y_{n+1} = g(t_{n+1}) + h(1-\theta) \sum_{j=0}^{n} k(t_{n+1}, t_j, y_j) + h\theta \sum_{j=1}^{n+1} k(t_{n+1}, t_j, y_j),$$

$n=0, 1, \ldots, \theta \in [0, 1]$. This method is reducible to the θ-method for ODEs with

$$\rho(\xi)=\xi-1, \qquad \sigma(\xi)=\theta\xi+(1-\theta).$$

For $\theta \in [0, \frac{1}{2})$ this method is explicit and as shown in [1] the region of absolute stability is empty. It follows from Theorem 3 that for $\theta \in (\frac{1}{2}, 1]$ the region of absolute stability is $\mathscr{A}=\mathscr{B} \cup \mathscr{C}$, where $\mathscr{B}$ and $\mathscr{C}$ are defined above. This is an improvement over the result given in [1]. For $\theta=\frac{1}{2}$ (this corresponds to the trapezoidal rule), the region of absolute stability is $\mathscr{A}=\mathscr{B}^* \cup \mathscr{C}^*$, where

$$\mathscr{B}^*=\{(h\lambda, h^2\mu, h^2\nu): 2\mu+\nu<0,\ \mu+\nu<0,\ h^2\nu<4\},$$

$$\mathscr{C}^*=\{(h\lambda, h^2\mu, h^2\nu): 2\mu+\nu>0,\ \mu+\nu>0,\ h^2\nu>4\}.$$

The same result was obtained in [1].

References

1. V. L. Bakke and Z. Jackiewicz, "Stability of Reducible Quadrature Methods for Volterra Integral Equations of the Second Kind," Numerical Analysis Technical Report 65R20-2, University of Arkansas, Fayetteville, December 1983; *Numer. Math.* **47** (1985), 159–173.
2. H. Brunner, A survey of recent advances in the numerical treatment of Volterra integral and integro-differential equations, *J. Comput. Appl. Math.* **8** (1982), 213–229.
3. H. Brunner and J. D. Lambert, Stability of numerical methods for Volterra integro-differential equations, *Computing* **12** (1974), 75–89.
4. E. W. Cheney, "Introduction to Approximation Theory," McGraw-Hill, New York, 1966.
5. C. V. Coffman, Asymptotic behavior of solutions of ordinary difference equations, *Trans. Amer. Math. Soc.* **110** (1964), 22–51.
6. G. Dahlquist, Convergence and stability in the numerical integration of ordinary differential equations, *Math. Scand.* **4** (1956), 33–53.
7. M. Evgrafov, The asymptotic behavior of solutions of difference equations, *Dokl. Akad. Nauk SSSR* **121** (1958), 26–29. [Russian]
8. P. Henrici, "Discrete Variable Methods in Ordinary Differential Equations," Wiley, New York, 1962.
9. J. Matthys, *A*-stable multistep methods for Volterra integro-differential equations, *Numer. Math.* **27** (1976), 85–94.
10. J. J. H. Miller, On the location of zeros of certain classes of polynomials with applications to numerical analysis, *J. Inst. Math. Appl.* **8** (1971), 397–406.
11. L. M. Milne-Thomson, "The Calculus of Finite Differences," Chelsea, New York, 1981.
12. O. Perron, Über Summengleichungen und Poincarésche Differenzengleichungen, *Math. Ann.* **84** (1921), 1–15.
13. O. Perron, Über Stabilität und asymptotisches Verhalten der Lösungen eines Systems endlicher Differenzengleichungen, *J. Reine Angew. Math.* **161** (1929), 41–64.
14. H. Poincaré, Sur les équations linéaires aux différentielles ordinaires et aux différences finies, *Amer. J. Math.* **7** (1885), 203–258.

15. P. J. van der Houwen and P. H. M. Wolkenfelt, On the stability of multistep formulas for Volterra integral equations of the second kind, *Computing* **24** (1980), 341–347.
16. P. H. M. Wolkenfelt, On the numerical stability of reducible quadrature methods for second kind Volterra integral equations, *Z. Angew. Math. Mech.* **61** (1981), 399–401.
17. P. H. M. Wolkenfelt, The construction of reducible quadrature rules for Volterra integral and integro-differential equations, *IMA J. Numer. Anal.* **2** (1982), 131–152.
18. P. H. M. Wolkenfelt, On the relation between the repetition factor and numerical stability of direct quadrature methods for second kind Volterra integral equations, *SIAM J. Numer. Anal.* **20** (1983), 1049–1061.